BPS-spectra in four dimensions from M-theory

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Abstract: I review my work together with Piljin Yi [1] on the spectrum of BPS-saturated states in $N = 2$ supersymmetric Yang-Mills theories. In an M-theory description, such states are realized as certain two-brane configurations. We first show how the central charge of the $N = 2$ algebra arises from the two-form central charge of the eleven-dimensional supersymmetry algebra, and derive the condition for a two-brane configuration to be BPS-saturated. We then discuss how the topology of the two-brane determines the type of BPS-multiplet. Finally, we discuss the phenomenon of marginal stability and show how it is related to the mutual non-locality of states.

1 Introduction

A recurring theme in many of the developments in string theory and supersymmetric field theory over the last few years has been the study of the spectrum of BPS-saturated states. The best known example is probably theories with $N = 2$ extended supersymmetry in four-dimensional Minkowski space $\mathbf{R}^{1,3}$, where the supertranslations algebra reads

$$\{\eta_{\alpha a}, \bar{\eta}_{\beta b}\} = \delta_{ab}(\gamma^\mu)_{\alpha\beta} P_\mu + \epsilon_{ab} \left(\Pi^+_{\alpha\beta} Z + \Pi^-_{\alpha\beta} \bar{Z} \right). \tag{1}$$

The supercharges $\eta_{\alpha a}$, $a = 1, 2$ are Majorana spinors and transform as a doublet under the $SU(2)_R$ symmetry of the $N = 2$ algebra. P_μ is the momentum and Z is the complex central charge. The projection operators on positive and negative chirality are denoted $\Pi^\pm_{\alpha\beta}$.

The central charge Z is in general a linear combination of conserved Abelian charges, such as for example electric and magnetic charges n_e and n_m and quark number charges S, i.e.

$$Z = a \cdot n_e + a_D \cdot n_m + m \cdot S. \tag{2}$$

The dependence of the coefficients a, a_D and m on the parameters and the moduli of the theory can often be described by Seiberg-Witten theory [2]. This amounts to introducing an auxiliary Riemann surface Σ on which a closed meromorphic differential λ is defined. To a set of quantum numbers n_e, n_m and S

[1] Talk presented at the workshop 'Quantum aspects of gauge theories, supersymmetry and unification, Neuchâtel University, 18-23 September 1997, to appear in the proceedings.

hep-th/9711076
CERN-TH/97-319
November 1997

corresponds a homology class $[C]$ on Σ, and the central charge in this sector is then given by the integral of λ along a representative C of $[C]$:

$$Z = \int_C \lambda. \tag{3}$$

Furthermore, the intersection number $[C] \cdot [C']$ of two homology classes $[C]$ and $[C']$ has an interesting interpretation. It is given by

$$[C] \cdot [C'] = n_e n'_m - n_m n'_e. \tag{4}$$

In a first-quantized treatment of a particle with charges corresponding to the homology class $[C']$ in a background containing a particle with charges corresponding to the homology class $[C]$, the 'wave-function' is really a section of the line bundle L over $\mathbf{R}^3 - \{0\}$, whose Chern number $c_1(L)$ equals $[C] \cdot [C']$. A non-zero intersection number thus means that L is topologically non-trivial. The physical interpretation is that these particles then are mutually non-local, such as for example an electron and a magnetic monopole.

The mass M of a unitary representation of the $N = 2$ algebra must obey the BPS-bound

$$M \geq |Z|. \tag{5}$$

Representations that saturate this bound are of particular interest [3]. If we limit ourselves to spin ≤ 1, they are either vectormultiplets (i.e. a gauge boson and its superpartners) or hypermultiplets (i.e. a quark and its superpartners). The BPS-saturated states enjoy an important stability property: Generically, such a state cannot decay, simply because it has the minimal mass in its charge sector. The exception is at a domain wall of marginal stability in the moduli space of vacua, where the phases of the central charges of three BPS-saturated states are equal. It might then be possible for the heaviest particle to decay into the two lighter ones (provided of course that the quantum numbers are conserved). The heaviest particle would then be absent from the spectrum on the other side of the domain wall of marginal stability.

To get a simple example of marginal stability, we can consider the case of a hypermultiplet quark in the background of a hypermultiplet dyon at weak coupling. The fermionic component of the hypermultiplet may have normalizable modes, each of which gives rise to a double degeneracy of dyonic states [4]. If the normalizable fermion mode disappears somewhere in the moduli space, so does the extra dyonic state. The interpretation is that it has experienced a marginal decay into the other dyonic state and a quark. The number of normalizable fermion modes is given by an index theorem, and in particular one can calculate the number of such modes that appear or disappear as the domain wall of marginal stability is crossed [5]. The result equals the Chern number of the 'wave-function' line bundle for the quark in the dyon background that we discussed above. Thus, at least in this case, a BPS-saturated particle can only experience marginal decay into two other particles if these are mutually non-local.

We would like to get a more general understanding of exactly when the phenomenon of marginal stability really does occur, or, more generally, what is the spectrum of BPS-saturated states at a given point in the moduli-space? We will address this problem in the context of an M-theory description of these models, following [6]. (Other approaches are described in [7][8].)

2 The M-theory realization

We consider M-theory on an eleven-manifold which is a direct product of four-dimensional Minkowski space $\mathbf{R}^{1,3}$ and some seven manifold X^7. We also include a five-brane, the world-volume of which fills $\mathbf{R}^{1,3}$ and defines a two-dimensional compact surface Σ in X^7 so that space-time Poincaré invariance is unbroken. Our four-dimensional field-theory will then be the infrared limit of the world-volume theory on the five-brane.

Unbroken supersymmetries are generated by spinor fields that

i) are covariantly constant with respect to the background metric.

ii) have positive chirality with respect to the tangent space of each five-brane world-volume element.

To get an $N = 2$ theory in $\mathbf{R}^{1,3}$, we must thus impose certain restrictions on the background metric and the five-brane configuration: First of all, we will take X^7 to be a manifold of $SU(2)$ holonomy. In fact, this means that X^7 is a direct product of $\mathbf{R}^3$ and a four-manifold Q^4 of $SU(2)$ holonomy, i.e. the eleven-manifold on which M-theory is defined is of the form

$$M^{1,10} \simeq \mathbf{R}^{1,3} \times \mathbf{R}^3 \times Q^4, \tag{6}$$

(The double cover of) the Lorentz group of $\mathbf{R}^3$ will then constitute the $SU(2)_R$ symmetry of the $N = 2$ algebra. The $SU(2)$ holonomy of Q^4 means that this four-manifold is a hyper-Kähler manifold, i.e. it admits a two-sphere S^2 of inequivalent complex structures J. Equivalently, Q^4 can be described as a Ricci-flat Kähler manifold, and therefore admits a covariantly constant holomorphic two-form Ω. The relationship between these two descriptions is as follows: Given a choice of complex structure J, i.e. a point on the two-sphere S^2, we have $\Omega \sim K' + iK''$, where K' and K'' are the Kähler forms corresponding to two other complex structures J' and J'' such that J, J' and J'' are all orthogonal.

To leave the $SU(2)_R$ symmetry unbroken, the five-brane world-volume must lie at a single point p in $\mathbf{R}^3$ and define some two-dimensional surface Σ in Q^4. To preserve $N = 2$ supersymmetry in $\mathbf{R}^{1,3}$, Σ must be a supersymmetric surface, which in the case under consideration means that it must be holomorphically embedded with respect to some complex structure J on Q^4. The unbroken supersymmetry generators are then of the form

$$\eta_{\alpha a} = \Pi^+_{\alpha\beta}\zeta^+_i Q_{\beta a i} + \Pi^-_{\alpha\beta}\epsilon_{ab}\zeta^-_i Q_{\beta b i}. \tag{7}$$

Here $Q_{\beta b i}$ are the supercharges of M-theory with the 11-dimensional spinor index decomposed into spinor indices for $SO(1,3)$, $SO(3)$ and $SO(4)$. $\zeta^\pm_i$ are the covariantly constant spinors on Q^4 whose existence follows from the $SU(2)$ holonomy. They have been chosen to have definite and opposite chirality with respect to the tangent space of each element of Σ. This is possible because Σ is holomorphically embedded with respect to the complex structure J on Q^4. The $\eta_{\alpha a}$ thus have positive chirality with respect to the five-brane world-volume. (The $\Pi^\pm_{\alpha\beta}$ project on definite and opposite chirality in $\mathbf{R}^{1,3}$.) They are Majorana spinors in $\mathbf{R}^{1,3}$ because the $Q_{\beta b i}$ are Majorana spinors in $M^{1,10}$ and ζ^+_i and ζ^-_i are each others charge conjugates on Q^4. (ϵ_{ab} is the charge conjugation matrix on $\mathbf{R}^3$.)

We have described a particular vacuum state of an $N = 2$ theory in $\mathbf{R}^{1,3}$ in terms of a background metric and a five-brane configuration. We now wish to describe excitations around this vacuum in terms of a two-brane, the world-volume of which traces a world-line in $\mathbf{R}^{1,3}$, lies at a single point p in $\mathbf{R}^3$ and defines a two-dimensional surface D in Q^4. The mass M of such a two-brane is proportional to the area of D, so to get a state of finite mass, we will consider an open surface D, whose boundary $C = \partial D$ lies on Σ. This means that the two-brane ends on the five-brane [9]. The quantum numbers of the corresponding state are then given by the homology class $[C]$ of C on Σ.

It now follows from the 11-dimensional supertranslations algebra in the presence of a two-brane [10]

$$\{Q_A, \bar{Q}_B\} = (\Gamma^M)_{AB} P_M + (\Gamma^{MN})_{AB} Z_{MN}, \tag{8}$$

where the two-form central charge Z_{MN} is given by

$$Z^{MN} \sim \int_D dX^M \wedge dX^N \tag{9}$$

that the supercharges (7) fulfill the $N = 2$ algebra (1) in $\mathbf{R}^{1,3}$ with the central charge Z given by

$$Z = \int_D \Omega_D. \tag{10}$$

Here Ω_D is the pullback to D of the two-form $\Omega = \bar{\zeta}^- \gamma_{st} \zeta^+ dX^s \wedge dX^t$. As the notation suggests, Ω is in fact the covariantly constant holomorphic two-form on Q^4. Provided that Ω is exact in a neighborhood of D, i.e. that $\Omega = d\lambda$ for some one-form λ, we can use Stokes' theorem to recover the Seiberg-Witten expression (3) for the central charge as the integral of λ along C.

3 The BPS-saturated spectrum

We have seen that the mass M and the central charge Z of the two-brane are given by the area of the surface D and the integral over D of the holomorphic two-form Ω respectively. A short calculation shows that the BPS-bound for the mass is saturated if and only if the pullback of the Kähler form K to D vanishes identically and the phase of the Hodge dual (with respect to the induced metric on D) of Ω is a constant. The second condition in fact means that there is a second Kähler form K'' whose pullback to D vanishes. This implies that the surface D is holomorphically embedded with respect to the complex structure J' which is orthogonal to the complex structures J and J'' corresponding to K and K''. Note that given the complex structure J with respect to which the surface Σ describing the five-brane is holomorphically embedded, there is a circle S^1 of orthogonal complex structures J' corresponding to the phase of the central charge Z.

Given a set of quantum numbers n_e, n_m and S, i.e. a homology class $[C]$ on Σ, we can calculate the corresponding central charge Z. Its phase determines a complex structure J' on Q^4, and a BPS-state with these quantum numbers corresponds to a surface D which is holomorphically embedded with respect to J' and whose boundary $C = \partial D$ lies on Σ and represents the homology class $[C]$. The problem of determining the BPS-saturated spectrum is thus equivalent to the problem of determining for which homology classes $[C]$ such a surface D exists. The construction of such a D is facilitated by the fact that its bulk behavior is completely determined by its boundary C by analytic continuation. On the other hand one should note, however, that a generic surface D only intersects the surface Σ at isolated points, and not along a curve. Also, a D which does intersect Σ along a closed curve C might run off to infinity somewhere in the bulk. Given $[C]$, it is therefore in general a non-trivial problem to establish the existence of a corresponding surface D.

In certain cases, and for special values of the moduli, it is possible to find exact solutions for such surfaces D. One can also do numerical studies. Comparison with known results in $N = 2$ super Yang-Mills theory led us to formulate the following conjecture (see also [11]):

BPS-saturated hypermultiplets and vectormultiplets correspond to surfaces D with the topology of a disc or a cylinder respectively.

Surfaces of more complicated topology do not seem to arise for generic values of the moduli, corresponding to the absence of other BPS-saturated multiplets of spin ≤ 1.

Since a BPS-saturated state cannot decay at a generic point in moduli space, it must in general be possible to accommodate an infinitesimal deformation of the surface Σ describing the five-brane by an infinitesimal deformation of the surface D describing the two-brane so that it is still holomorphically embedded and ends on Σ. However, a BPS-saturated state can decay into two other BPS-saturated states of appropriate quantum numbers at a domain wall of marginal stability where the three central charges have the same phase. This must mean that the surface D of the heaviest particle degenerates into two surfaces touching in isolated points and does not exist on the other side of the domain wall of marginal stability.

In this way, a hypermultiplet can decay into two other hypermultiplets when the corresponding disc degenerates into two discs touching in a point. The intersection number $[C] \cdot [C']$ of the corresponding homology classes thus equals ± 1. For example, a quark-dyon bound state in $SU(2)$ Yang-Mills theory with fundamental flavors can decay into a dyon with $(n_e, n_m) = (2n, 1)$ and a quark with $(n'_e, n'_m) = (1, 0)$ so that indeed $n_m n'_e - n_e n'_m = 1$. Similarly, a vectormultiplet can decay into two hypermultiplets when the corresponding cylinder degenerates into two discs touching in two points, i.e. the intersection number must equal ± 2. For example, a W-boson in $SU(2)$ Yang-Mills theory can decay into two dyons with $(n_e, n_m) = (2n, 1)$ and $(n'_e, n'_m) = (2 - 2n, -1)$ so that indeed $n_m n'_e - n_e n'_m = 2$. We see that in both cases, the mutual non-locality of the constituent states is a necessary (and possibly sufficient) requirement for the decay to take place.

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